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The quartic curve and its inscribed configurations

by H. Bateman.

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§1. Introduction

Whereas the geometry of a planar cubic curve can be regarded as fairly complete, that of the quartic is far from being so. It is true that our present knowledge of the properties of the curve is very extensive, as may be seen from the admirable article by G. Kohn and G. Loria in the Encyklopädie der Mathematischen Wissenschaften[1]; but there are several important questions which have still to be answered. some of these are mentioned in Ciani's recent report[2].

At present the most useful form for the equation of a general quartic is that obtained by regarding the curve as the envelope of a family of conics[3]

$$\lambda^2 S_0 + 2\lambda S_1 + S_2 = 0,$$

where λ is an arbitrary parameter. This form, however,

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1. Band. III. 2. Heft. 4. (1909). pp. 514 – 540

2. Le curve piane di quart'ordine. Giornale di Matematiche, t. 48 (1910). pp 259 – 304.

3. G. Salmon. Higher plane curves. Dublin. 1852. p 196

is unsuitable for a discussion of the invariants and covariants of the curve. A canonical form consisting of the sum of the fourth powers of five linear functions $u_1, \ldots, u_5$ is used for this purpose for Clebsch[1] has shown that the equation of a quartic curve can be thrown into this form only when the invariant B [illegible] number.

The sum of six fourth powers is a possible form[2] but the equation of the general quartic can be reduced to this form in ∞^3 ways. It is easy to deduce from this equation that three corners of the hexagon given by the six linear forms form an apolar triad with regard to the quartic and that the sides of two such hexagons touch a curve of the third class. but the equation is not adapted for a simple discussion of other geometrical

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1. Crelle's Journal. Bd 59. (1861). p. 125.

2. Rosanes. Crelle's Journal. Bd. 76. (1873). p. 329.

Scherrer. Progr. Franenfeld. p. 17.

Scorza. Annali di Matematica (3). t. 2 (1899) p. 329.

properties of the curve.

It has been found by experience that it is convenient to have a number of typical forms for the equation of the curve, each form being appropriate for the study of the properties of the curve relative to some inscribed configuration. In some cases the existence of the configuration implies that the quartic curve is not general. Such cases are studied here in detail so as to prepare the way for the determination of the relations between the invariants which correspond to each particular case.

A complete system of rational covariants of the general quartic curve has not yet been obtained in a form which is easy to use, but thanks to the labours of Salmon[1], Clebsch[2], Maisano[3], Gordan[4], Caporali[5], Pascal[6] and Emmy

1. Higher Plane Curves 2. Loc cit.

3. Giornale di Matematiche, t. 19. (1881). p. 198 4 Math. Ann. Bd 20 (1882) p. 483

5. Memorie di Geometria, Naples. (1888).

6. Napoli Atti (2) t 12 (1905). p. 1.

Noether[1], considerable progress has been made. In particular, Pascal has obtained criteria for certain types of degeneration and Noether has constructed a 'relatively complete system of forms'. Dr^s Morley and Conner[2] have found the relation which connects the invariants A and B when the curve contains ∞^1 configurations of fifteen points lying by threes on twenty lines, while Caporali has obtained some of the relations which are [illegible] certain other special types of curve.

A result of considerable importance for the invariant theory has been obtained by writers on Abelian functions of genus three[3]. It follows from the work of Riemann and Schottky that the invariants of a quartic curve can be

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.1. Crelle's Journal Bd. 134) (1908).

.2. American Journal. Vol. 31. p. 263.

.3. B. Riemann, Werke, 2nd ed^n, p. 427 H. Weber, Theorie der Abelschen Functionen vom Geschlecht drei, Berlin (1876); F. Schottky, Abriss einer Theorie der Abelschen Functionen von drei Variabeln, Leipzig 1880, also Acta Mathematica, t. 27. (1903), & Berlin. Berichte (1910).

expressed rationally in terms of six fundamental quantities which can be regarded as irrational invariants of the curve. In particular, the twenty eight bitangents can be represented by equations in which the coefficients are rational functions of the six fundamental quantities.[1] Cayley has shown that the quartic curve possesses a double point when the irrational invariants are connected by a certain relation. The Abelian ... has been further developed by Klein,[2] Wiltheiss[3], Baker[4] and J. E. Wright.[5]

Frobenius[6] has obtained similar results by algebraic methods. In his second memoir he starts with Hesse's method[7] of deriving a general quartic curve from a net of quadric surfaces

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1. See also. A. Cayley Collected Papers. Vol. XI. p. 221, Vol. XII. p. 74. R. de. Paolis Mem. Lincei (3) 2. (1878).

2. Math. Ann Bd 10. (1876)

3 Untersuchungen über Thetafunktionen. Leipzig (1895) Math. Ann. Bd 40

4 Multiply periodic functions. Cambridge (1907). 5. Amer. Journ. Vol 31. (1909)

6 Crelle's Journal Bd 99 (1886). Bd. 103, (1888). 7. Ibid Bd 49 (1855)

touching eight associated planes. He uses the symbol $f_{\alpha\beta\gamma\delta}$ to denote the determinant formed from the homogeneous coordinates of four of these planes and shows that the 35 ratios $f_{\lambda\mu\nu\rho} : f_{0123}$ can be expressed rationally in terms of the sixteen ratios $f_{\lambda\beta\gamma\delta} : f_{0123}$ $\lambda = 4, 5, 6, 7$, in which the suffixes β, γ, δ can denote any three of the numbers 0, 1, 2, 3 and the numbers 0, 1, 2, ... 7 are used to denote the eight associated planes. Now these sixteen ratios form an orthogonal matrix on account of the existence of relations of the type

$$\sum f_{\lambda\alpha\beta\gamma} f_{\lambda\alpha'\beta'\gamma'} = 0 . \qquad \begin{matrix} \lambda = 4, 5, 6, 7 \\ \alpha \neq \alpha', \ \beta \neq \beta', \ \gamma \neq \gamma' \end{matrix}$$

and can consequently be expressed rationally in terms of six fundamental quantities with the aid of Cayley's formulae for the coefficients of an orthogonal linear transformation[1].

Weber recognizes that it is more convenient to retain the quantities $f_{\alpha\beta\gamma\delta} : f_{0123}$ in spite of the relations between them, and to regard these as the fundamental irrational invariants of the quartic. He shows in particular that if $\mathfrak{C}_{\alpha\beta}$ $(\alpha, \beta = 0, 1, \ldots 7)$ are 28 linear forms which when equated to zero give the

1. Crelle's Journal, Bd 32 (1846), p. . Collected Papers, Vol. 2, p.

equations of [illegible] bitangents [illegible] satisfies [illegible]

$$\sum_{\lambda}{}' \int_{\beta\gamma\delta\lambda} x_{\alpha\lambda} \equiv 0 .$$

connecting the forms belonging to four bitangents of an Aronhold system. The summation extends over the values of λ which differ from $\alpha.\beta.\gamma.\delta$.

An important consequence of this result is that if we know the equations of seven bitangents of an Aronhold system we can determine the irrational invariants.

Now the [illegible] of a quartic curve in relation to an Aronhold system of bitangents may be discussed very conveniently with the aid of a (1.2) 'transformation' in which the lines of a plane X' correspond to a net of cubics through seven fixed points in a plane X. A point P' in X' consequently corresponds to a pair of points $P_1.P_2$ in X and when these come together P' lies on a quartic curve which is called the [illegible] curve [illegible]

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1 A. Clebsch. Math. Ann. Vol 3. (1871) M. Noether Erlangen Berichte 10. (1878). München. Abh 17. (1889). R. de Paolis Loc. cit. G. Frobenius. Loc. cit

2. S. Aronhold. Berlin Berichte (1864)

(Übergangskurve)[1] of the transformation. The seven base points in the plane X correspond to seven bitangents of the limiting curve L and these bitangents form an Aronhold system. It is important to note that in many cases these bitangents can be derived from the seven base points by a correlation between the two planes and so the work of calculating Frobenius' irrational invariants is much simplified.

In this memoir we shall endeavour to prepare the way for a discussion of the conditions that a quartic curve may be of a particular type by looking for a (1.2) transformation which has the species of quartic curve as limiting curve. The appropriate transformation has already been found in a number of cases. It is known for instance that the quartic curve L has a double point[2] when three of the base points lie on a line or six on a conic. L consists of two conics[3] when six of

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.1. Also called the synoptic curve. Miss C.A. Scott Quarterly Journal. Vol. 29.

.2. De. Paolis. Loc. cit

.3. This follows at once from the known representation of a rational curve (with four nodes) on a quadric

the base points are at the corners of a complete quadrilateral. L consists of four straight lines[1] when six of the base points are consecutive in pairs and the lines joining the three pairs meet at the seventh base point. The conditions on the seven points usually take two or more alternative forms for it should be noticed that the limiting curve L is unaltered when the net of curves in the plane X is transformed into another by a quadratic transformation with base points at three of the seven points.

We shall prove in this memoir that the Lüroth quartic arises from a (1,2) transformation in which the seven base points have the same polar lines with regard to a conic and a cubic. When the cubic breaks up into three straight lines the desmic quartic is obtained thus gives a simple verification of Cayley's theorem[2] that the desmic quartic is a particular case of Lüroth's quartic.

The known fact that Lüroth's quartic can be described

.1. H. Bateman. Quarterly Journal. (1906). American Journal (1913).

.2. Crelle's Journal. Bd. 95. (1883).

from eight associated points which are the poles of a plane with regard to a cubic surface[1] is next used to obtain a construction for seven points which have the same polar lines with regard to a conic and cubic curve. The desmic quartic arises when the plane meets the cubic surface in three lines.

A (1.2) transformation is next set up by mapping the chords of a twisted cubic in space on the points of a plane. The two points in which a chord meets a fixed quadric then correspond to the point in the plane which is associated with the chord. The (1.2) transformation between two planes is finally obtained by mapping the quadric surface on a plane by stereographic projection, using one of the points in which the quadric is met by the twisted cubic as vertex of projection.

A transformation which gives rise to a desmic quartic is obtained in this way. The representation also leads to a notable property of a quadratic complex of lines which contains all the lines joining five points in space. It also

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1. W. Fraham, Math Ann. Bd. 7 (1874), E. Toeplitz, Math. Ann. Bd. 11 (1877)

leads to the consideration of a type of uninodal quartic containing ∞^1 configurations of ten points lying by threes on ten lines and to a new proof of the theorem that a plane section of Weddle's surface contains ∞^1 configurations of fifteen points lying by threes on twenty lines.

The method due to E. Godt[2] and E. Timerding[3] of deriving a quartic curve, and in fact a (1.2) transformation, from a general quadratic transformation between the lines of a plane is next studied and quadratic transformations are found which give rise to the Lüroth quartic and desmic quartic respectively.

Klein's quartic is finally obtained as the limiting curve of a (1.2) transformation and a set of eight associated points in space, from which the curve can be derived, is deduced from this result. The known equations of the 28

1. H Bateman. Proc. London Math Soc. Ser. 2. Vol. 3. (1905). Morley and Conner. Loc. cit.

2. Diss. Göttingen (1893) Clebsch - Lindemann. Vorlesungen. p. 1004.

3. Math. Ann. Bd. 53. (1900). p. 193

bitangents of Klein's quartic are obtained by a simple method.

Dr Coble[1] has extended the known reduction of the equations of a point conic and line conic to the sums of three squares, which when equated to zero give three lines and three points respectively, which are polars of one another with regard to both conics. His extension relates to a point quartic and a line quartic and two sets of six conics. Now it seems natural to try and extend other simplified forms of the equation of a conic to a quartic curve. An extension of the equation of a conic referred to a circumscribed triangle leads to the known form

$$\sqrt{x_1 x_2} + \sqrt{y_1 y_2} + \sqrt{z_1 z_2} = 0 .$$

of the equation of the general quartic curve. The corresponding generalisation of the equation of a conic referred to an inscribed triangle leads to the problem of reducing the equation of the general quartic to the form

$$y_1 y_2 z_1 z_2 + z_1 z_2 x_1 x_2 + x_1 x_2 y_1 y_2 = 0 .$$

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1. American. Transactions. Vol. 4 (1903).

This problem is shown to be equivalent to that of finding a triangle which is inscribed in a given cubic and circumscribed to a given conic and in general this problem possesses a limited number of solutions. In a special case, however, ∞^1 quadrilaterals can be circumscribed to the conic and completely inscribed in the cubic. I have shown that in this case the corresponding quartic curve is a desmic quartic. I did not recognise this at first and was consequently led to rediscover some of the known properties of the desmic curve. My excuse for discussing these at length is that the present analytical treatment is [illegible] applies to the discussion of the properties of Hesse's configuration.

A description of this configuration is useful at this point because it indicates clearly the properties of the ∞^1 sets of 24 points which can be inscribed in Caporali's quartic.

Such a set of 24 points can be divided into two sets of three quartets such that any two quartets belonging to different sets lie on a pair of lines. Now I have discovered a second type of quartic curve containing ∞^1

sets of 24 points possessing this same property, but curiously enough the configuration of 24 points is distinct in character from the set of 24 points which arises from Hesse's configura-

I have been led in the course of these researches to a few miscellaneous theorems which are given at the end of the dissertation. A set of eleven points in a space of four dimensions has been found with the property that each point is the vertex of a quadric cone passing through the other ten. A new property of a four-nodal cubic surface has been discovered and a diagram is drawn showing the symbols for the twenty eight bitangents of a quartic curve in Hesse's notation.

I wish to thank Professors Morley and Coble for the encouragement and helpful suggestions they have given me during the preparation of this thesis.

§2. Lüroth's quartic

Lüroth[1] has shown that a quartic curve which passes through all the intersections of five lines is not general, for it is the covariant S of a quartic of Clebsch's type i.e. a quartic whose equation can be expressed as the sum of five fourth powers, and since this type of quartic depends on only thirteen constants it follows that the equation of Lüroth's quartic likewise involves only thirteen independent constants.

Lüroth has shown moreover that when one pentagon is known to be completely inscribed in a given quartic curve there are ∞ pentagons possessing the same property and their sides all touch a conic.[3]

A particular (1.2) transformation which has the Lüroth quartic as [illegible] obtained as [illegible]

1. Math. Ann. Bd. 1. (1869) Bd 15 (1878).

2. I use the word pentagon here to mean the figure formed by five lines.

3. See also Darboux. Sur une classe remarquable de courbes et de surfaces algébriques. Paris. (1896). p. 126. W.H. (Thirteen) [illegible] Dupuis

Let a point P correspond to the two points Q, Q' in which the polar ~~plane~~ of P with regard to a conic C_2 meets the polar conic of P with respect to a cubic C_3. There is evidently a (1,2) correspondence between P and Q for when Q is given, P is the point of intersection of the polar lines of Q with regard to the conic and cubic.

To find the locus of P when Q and Q' come together we take the equations of the cubic and conic in the forms

$$(ax^3) = 0 \quad , \qquad (bx^2) = 0 \quad , \qquad (x) \equiv 0 \, .$$

where the symbol (ax^3) is used to denote the sum of four terms with different suffixes. The four lines $x_1 x_2 x_3 x_4 = 0$ are the common tangents of the ∞' conics apolar to C_2 and C_3, they form a quadrilateral whose six corners lie on the Hessian of C_3.

Let (y_1, y_2, y_3, y_4) be the coordinates of P. Its polar with regard to C_3 [illegible]

1. A reduction to these forms is given by R.A. Roberts, Proc. London Math. Soc. (1) Vol 21. p 62.

$(b_{yz}\,\infty) = 0$

and its polar conic with regard to $(')_5$ is

$a_{yz}\,\infty^2) = 0$

When Q and Q' come together the line QQ' touches this polar conic at the point Q $(x_1 x_2 x_3 x_4)$ and so its equation must be equivalent to

$$(a_{yz}\,\infty) = 0 .$$

Since $(\infty) \equiv 0$ we must have a set of relations of the type

$$a_{rs} y_s z_r = \lambda b_{rs} y_s + \mu \qquad (r = 1,2,3,4)$$

Using the relations $(x) = 0$, $(b_{yz}) = 0$, we find that

$$\lambda \left(\frac{b}{a}\right) + \mu \left(\frac{1}{a_{yz}}\right) = 0 ,$$

$$\lambda \left(\frac{b^2 y}{a}\right) + \mu \left(\frac{b}{a}\right) = 0 .$$

For [illegible] $= \left(\frac{b}{a}\right)^2$

[illegible] $+ \left(\frac{b}{a}\right)^2 a_{rs} y_s = 0$

[illegible] with different suffixes — see that the [illegible] which Q and Q' come together [illegible]

quartic $\left[\frac{1}{a_{ij}}\right] = 0$

Since the equation of the Hessian of C_3 is $\left(\frac{1}{a_{ij}}\right) = 0$ it appears that the Hessian H_3 touches the Lüroth quartic L at the six corners of the quadrilateral.

This indicates a method by which a suitable C_3 can be determined when L is given. Since there are ∞' pentagons completely inscribed in L it appears that there are ∞' suitable C_3's.

Returning to the transformation we notice that if P describes a straight line

$$(\xi y) = 0,$$

the two corresponding points Q describe a cubic curve whose equation is obtained by eliminating $y_1\ y_2\ y_3\ y_4$ from the four equations

$$(y) = 0\ ,\quad (\xi y) = 0\ ,\quad (b_{ijk}x) = 0\ ,\quad (a_{ijk}x^2) = 0$$

It is easy to see that the equation of the cubic can be derived from that of the line by writing

$$y_1 = a_3b_3\,x_2x_3^2 - a_2b_3\,x_2^2x_3 + a_2b_4\,x_2^2x_4 - a_4b_2\,x_2x_4^2 + a_4b_3\,x_3x_4^2 - a_3b_4\,x_3^2x_4$$

$$y_2 = a_1b_3\,x_3x_1^2 - a_3b_1\,x_3^2x_1 + a_3b_4\,x_3^2x_4 - a_4b_3\,x_3x_4^2 + a_4b_1\,x_1x_4^2 - a_1b_4\,x_1^2x_4$$

$$y_3 = a_2b_1\,x_1x_2^2 - a_1b_2\,x_1^2x_2 + a_1b_4\,x_1^2x_4 - a_4b_1\,x_1x_4^2 + a_4b_2\,x_2x_4^2 - a_2b_4\,x_2^2x_4$$

$$y_4 = a_2b_3\,x_2^2x_3 - a_3b_2\,x_2x_3^2 + a_3b_1\,x_3^2x_1 - a_1b_3\,x_3x_1^2 + a_1b_2\,x_1^2x_2 - a_2b_1\,x_1x_2^2$$

and these are the equations of the transformation.

The cubic curve corresponding to a given line t can be generated by the points of intersection of corresponding members of a pencil of conics and a pencil of lines. The conics are the polar conics of points on t with regard to C_3 and so pass through the four poles of t with regard to C_3. The lines are the polars of points on t with respect to C_2, they all pass through T the pole of t with respect to C_2. This point T evidently lies on the cubic Γ_3 corresponding to t and the ~~po~~ four tangents from T to Γ_3 are the polars with respect to C_2 of the four points in which t meets L. Hence the invariant of Γ_3, i.e the cross ratio of the four tangents which can be drawn to it from any point on it, is equal to the cross ratio of the four points in which t meets L.

The four cubics $y_1=0$, $y_2=0$, $y_3=0$, $y_4=0$ all pass through the

seven points which have the same polar lines with regard to C_2 and C_3.

These seven points are the base points of the (1,2) transformation and their corresponding lines, viz their polars with regard to C_2 are by a well known theorem bitangents of the limiting curve L. Hence we have the following theorem.

The seven points which have the same polar lines with regard to a conic C_2 and a cubic C_3 are such that these polar lines form an Aronhold set of bitangents of the quartic.

It appears from this result that the seven points cannot be chosen arbitrarily, they depend on thirteen independent constants instead of fourteen. This would not be expected a priori because an arbitrary cubic and conic give fourteen arbitrary constants.

It is not easy to find the relation between the seven base points. At Prof Morley's suggestion I have considered the cubic C_3 and two points as given and found the locus of the other five.

Take the two given points and the intersection of their polar lines with regard to C_3 as corners of the triangle of reference. The equations of C_3 and C_2 then take the forms

$$ax^3+by^3+cz^3+3fy^2z+3gyz^2+3kzx^2+3lx^2y+6mxyz=0,$$

$$\lambda x^2 + by z^2 + cf x^2 + 2fgyzx = 0,$$

where λ is arbitrary. The possible conics C_2 all have double contact at two points on the line $x = 0$.

A point (x,y,z) has the same polar line with regard to C_3 and C_2 if the following equations are satisfied:-

$$\frac{ax^2 + \dots}{\lambda x} = \frac{lx^2 + by^2 + cz^2 + 2fyz + 2nzx}{bgy + fgz} = \frac{\dots}{\dots}$$

Hence it lies on the two cubic curves

$$f(gy+cz)(lx^2+by^2+gz^2+2fyz+2nzx) = g(by+fz)(kx^2+fy^2+cz^2+2gyz+2nxy)$$

$$\lambda x(lx^2+by^2+gz^2+2fyz+2nzx) = g(by+fz)(ax^2+2hxy+2kzx+2lxy).$$

The first equation does not contain λ and so represents the required locus when two points and C_3 are given. The locus is a cubic curve which circumscribes the triangle of reference and passes through the four poles of $x=0$ with regard to C_3. It also passes through the two points in which the polar conic of one of the given points meets its polar line. Since the term in x,y,z is absent from the equation it appears that the corners of the triangle of reference form a conjugate triad i.e the mixed polar of two points passes through the third.

It is clear that this cubic locus is the cubic T_3 corresponding to the line $x=0$ in the transformation. The associated point 'T' is the point $y=0, z=0$ and this point is coincident to the seven base points of the transformation.

If in the previous notation the equation of C_3 is

$$(ax^3) = 0, \qquad (x) \equiv 0,$$

and $(y_1y_2y_3y_4)$, $(z_1z_2z_3z_4)$ are the two given points, the equation of the locus is

$$(axy^2)(axz^2x)(ayz^2) = (axz^2 \; axz^2y)(ay^2x).$$

This equation was obtained by Prof. Morley.

Returning to the previous work we notice that as 'x' varies the second equation represents a pencil of cubics which pass through four fixed points on T_3, viz the two given points and the two points in which the polar conic of one of these points meets its polar line.

It appears that the different possible sets of 7 points associated with the two given ones and C_3 are cut out by a pencil of cubics through our fixed points on T_3 and so form a system on T_3.

§ 8 Derivation of the desmic quartic from a (1,2) transformation.

Let us now consider the case when the cubic Γ_3 consists of three straight lines. Taking these lines as sides of the reference triangle and using the equation of the conic in the form

$$(a,b,c,f,g,h)(x,y,z)^2 = 0 \,.$$

we find that the polar conic of a point $P\,(X,Y,Z)$ with regard to the cubic is

$$Xyz + Yzx + Zxy = 0 \,.$$

while the polar line of P with regard to the conic is

$$(aX+hY+gZ)x + (hX+bY+fZ)y + (gX+fY+cZ)z = 0$$

The line touches the conic if

$$\sqrt{X(aX+hY+gZ)} + \sqrt{Y(hX+bY+fZ)} + \sqrt{Z(gX+fY+cZ)} = 0$$

This, however is Humbert's equation for a desmic quartic curve[1]. It evidently represents a quartic curve having the lines $X=0$, $Y=0$, $Z=0$, $aX+hY+gZ=0$, $hX+bY+fZ=0$, $gX+fY+cZ=0$ as bitangents. Now where $X=0$ we have

$$bY^2 = cZ^2 \,.$$

hence it appears that the points of contact of the three bitangents $X=0$, $Y=0$, $Z=0$ are the corners of a complete quadrilateral whose sides are the lines

$$X\sqrt{a} \pm Y\sqrt{b} \pm Z\sqrt{c} = 0. \qquad (1)$$

Humbert has shown that there are altogether six triads of bitangents of the curve which possess this property. The lines $aX+hY+gZ=0$, $hX+bY+fZ=0$, $gX+fY+cZ=0$ form another of these triads.

It should be mentioned that since a, b, c, f, g, h are quite arbitrary the equation we have obtained represents the general dermic quartic and so we have the result that the dermic quartic is derived from a transformation associated with a conic C_2 and a cubic C_3 which consists of three straight lines.

This gives a simple verification of Schur's theorem that the dermic quartic is a particular case of Lüroth's quartic. This result may also be deduced from Humbert's remark that the sides of the quadrilateral formed by the lines (1) meet the quartic again in collinear points. This remark

[illegible] that there is more than one system of pentagons inscribed in the same quadrilateral — for if we can show that the quadrilaterals belonging to two triads of bitangents do not touch a conic we can be sure that the pentagons derived from these quadrilaterals by adding [illegible] different systems.

Now in the case of the two triads of bitangents whose equations have already been obtained, the sides of the two quadrilaterals are given by

$$X\sqrt{a} \pm Y\sqrt{b} \pm Z\sqrt{c} = 0.$$

$$(aX+hY+gZ)\sqrt{bc-f^2} \pm (hX+bY+fZ)\sqrt{ca-g^2} \pm (gX+fY+cZ)\sqrt{ab-h^2} = 0.$$

Conics touching the sides of these quadrilaterals have equations of the type

$$\lambda(aX+hY+gZ)^2 + \mu(hX+bY+fZ)^2 + \nu(gX+fY+cZ)^2 = 0.$$

respectively.

These equations are the same only if

$$\begin{aligned} \lambda gh + \mu bf + \nu fc &= 0 \\ \lambda gh + \mu fh + \nu cg &= 0 \\ \lambda ah + \mu hb + \nu gf &= 0. \end{aligned}$$

$$\text{if} \qquad g^2h^2(f^2-bc) + abfg(ch-fg) + cahf\,(bg-hf) = 0\,.$$

Since this equation is not generally satisfied we must infer that there are different systems of pentagons circumscribed with the two triads of bitangents. Now remembering that there is an equation of the form

$$\sqrt{X(aX+hY+gZ)} + \sqrt{Y(hX+bY+fZ)} + \sqrt{Z(gX+fY+cZ)} = 0\,,$$

associated with any two of the six triads of bitangents, hence we are led to the conclusion that there are at least six different systems of pentagons completely circumscribed to the Steiner quartic.

To determine the nature of the involution on the conic touching the sides of all the pentagons of one system we use Darboux's method.

Let $y^2 = \alpha x$ be the equation of the conic then the equations of the five sides of a circumscribed pentagon may be written in the form

$$\alpha_s^2 x + 2\alpha_s y + \alpha = 0\,, \qquad s = 0, 1, \ldots 4\,.$$

If now we put

$$x_s = \frac{\alpha_s^2 x + 2\alpha_s y + \alpha}{f'(\alpha_s)}$$

where [illegible]

the condition $(x) = 0$ is satisfied

The equation of a desmic quartic derived from a cubic C_3 consisting of the three lines $(x^3) = 0$, and the conic C_2 whose equation is $(bx^2) = 0$ is easily found to be

$$(b^2x)\left(\frac{1}{ax}\right) = (b)^2.$$

Taking $(b^2x) = 0$ to be the line (α_0) and noticing that (b) is arbitrary since any multiple of (x) can be added to (b^2x) without altering the equation of the quartic, we find that the equation takes the form

$$\sum_1^4 \frac{f'(\alpha_s)}{\alpha_s^2 x + 2\alpha_s y + z} + \frac{\lambda}{\alpha_0^2 x + 2\alpha_0 y + z} = 0,$$

where λ is the arbitrary constant $(b)^2$.

Introducing Darboux's coordinates

$$x : 2y : z = 1 : \theta+\phi : \theta\phi,$$

we find that the equation may be written in the form

$$\sum_1^4 \frac{f'(\alpha_s)}{\alpha_s - \theta} + \frac{\lambda}{\alpha_0 - \theta} = \sum_1^4 \frac{f'(\alpha_s)}{\alpha_s - \phi} + \frac{\lambda}{\alpha_0 - \phi}.$$

Hence the invariants of ... parameters of the ... quartic ... represented by an equation of the form

$$\sum_{1}^{4} \frac{f'(\alpha_0)}{\alpha_0 - \theta} + \frac{\lambda}{\alpha_0 - \theta} + \mu = 0$$

where μ is a variable parameter.

The dermic quartic is also obtained when the cubic C_3 and the conic C_2 are represented by the equations

$$x^3 + y^3 + z^3 + 6mxyz = 0$$

$$ax^2 + by^2 + cz^2 = 0 .$$

The polar conic and polar line of (X, Y, Z) are now

$$X(x^2 + 2myz) + Y(y^2 + 2mzx) + Z(z^2 + 2mxy) = 0 .$$

$$aXx + bYy + cZz = 0 .$$

respectively. The conic touches the line if

$$a^2X^2(YZ - m^2X^2) + b^2Y^2(ZX - m^2Y^2) + c^2Z^2(XY - m^2Z^2)$$

$$+ 2bc\,YZ(m^2YZ - mX^2) + 2ca\,ZX(m^2ZX - mY^2)$$

$$+ 2ab\,XY(m^2XY - mZ^2) = 0 .$$

$$= \left[(a^2 - 2mbc)X + (b^2 - 2mca)Y + (c^2 - 2mba)Z \right] XYZ$$

$$= m^2 \left[a^2X^4 + b^2Y^4 + c^2Z^4 - 2bcY^2Z^2 - 2caZ^2X^2 - 2abX^2Y^2 \right].$$

This however is one of Humbert's equations for a desmic quartic. The lines $X=0$, $Y=0$, $Z=0$ are bitangents whose points of contact lie at the corners of the quadrilateral formed by the four lines

$$\sqrt{a}X \pm \sqrt{b}Y \pm \sqrt{c}Z = 0$$

It appears then that if we want the $(1,2)$ transformation to give rise to a desmic quartic it is not necessary for the cubic C_3 to break up into lines L_{ij}

§ 4 Derivation of the (1,2) transformation from eight assigned points and vice versa.[1]

Let a point P whose coordinates are (X,Y,Z) relative to a fixed triangle in a plane E be made to correspond to a quadric surface whose equation is

$$\lambda S_0 + \mu S_1 + \nu S_2 = 0$$

where S_0, S_1, S_2 are of the second degree in the homogeneous coordinates (x,y,z,t). As P varies the quadric always passes through a set of eight associated points $A_0 A_1 \cdots A_7$.

Let the two generators through A_0 of the quadric corresponding to P meet a fixed plane M in two points Q, Q' and consider the (1,2) transformation by which P is derived from Q.

As P describes a straight line Q and Q' describe a cubic curve viz the projection of the biquadratic common to a pencil of quadrics through $A_0 A_1 \cdots A_7$. If $a_1 \cdots a_7$

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[1] The leading ideas of this section are due to Hesse, Clebsch and Frobenius. See the memoirs cited above.

are the projections of $A_1 \ldots A_7$ on the plane E when A_0 is taken as vertex of projection. the net of cubics through $a_1 \ldots a_7$ is the image of the net of lines in the plane E.

The points Q, Q' come together when the quadric corresponding to P is a cone and then by Hesse's theorem the locus of P is a quartic curve which is the limiting curve L of the $(1,2)$ transformation.

If the quadric (1) is always the polar quadric of P with regard to a cubic surface S_3, the eight associated points are the poles of the plane E with regard to S_3 and L is a plane section of the Hessian H_4 of the cubic surface. Now when the equation of S_3 is written in Sylvester's canonical form

$$(ax^3) = 0.$$

of the sum of five cubes. the equation of H_4 is

$$\left(\frac{1}{ax}\right) = 0$$

and it appears that any plane section is a [illegible] quartic. Hence we have the theorem

The fourth point can be formed by Hesse's method from eight associated points which are the poles of a plane with regard to a cubic surface.[1]

The condition to be satisfied in order that three quadrics through eight associated points may be polar quadrics of three points with regard to a cubic surface is discussed by Töplitz. Math. Ann. Bd. 11. The invariant Λ must vanish.[2] It appears that neither the cubic surface nor the [illegible] are uniquely determined when $\Lambda = 0$. The plane of the three poles and the five planes associated with the canonical form of S_3 osculate a twisted cubic.[3]

Dr. Coble has shown with the aid of Töplitz's invariant that when six of the eight points are given the other two lie on a quartic surface.

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1. W. Frahm. Math Ann. Bd. 7. (1874).

2. Cf. Salmon's Geometry of Three Dimensions 4th edition. pp 209-10

3. A. C. Dixon. Proc. London Math Soc. Ser. 2. Vol. 7. p 150.
Toeplitz. Loc. cit.

To deduce an equation for Lüroth's quartic from Fraken's result we take four poles of the plane E as vertices of the tetrahedron of reference. If now the equation of the cubic surface is

$$ax^3+by^3+cz^3+dt^3+3fy^2z+3gyz^2+3hz^2x+3kzx^2+3lx^2y+3mxy^2$$
$$+3ux^2t+3\alpha xt^2+3vy^2t+3\beta yt^2+3wz^2t+3\gamma zt^2+6pyzt$$
$$+6qzxt+6rxyt+6sxyz=0,$$

the polar plane of (x,y,z,t) is

$$X(ax^2+my^2+hz^2+\alpha t^2+2syz+2kzx+2lxy+2uxt+2vyt+2\beta zt)$$
$$+Y(lx^2+by^2+gz^2+\beta t^2+2fyz+2szx+2mxy+2rxt+2vyt+2pzt)$$
$$+Z(kx^2+fy^2+cz^2+\gamma t^2+2gyz+2hzx+2sxy+2qxt+2pyt+2wzt)$$
$$+T(ux^2+vy^2+wz^2+dt^2+2pyz+2qzx+2rxy+2\alpha xt+2\beta yt+2\gamma zt)=0$$

The polar planes of the corners of the reference tetrahedron coincide with the plane $X+Y+Z+T=0$. if

$$a-l-k=u, \quad m-f=v, \quad h-g=\gamma-w, \quad \alpha+\beta+\gamma=d$$

The equation of the polar quadric of an arbitrary point (X,Y,Z,T) of this plane is now found to be

$$X[(\gamma-p)yz+(\alpha-q)zx+(\alpha-r)xy+(\alpha-d)xt+(r-d)yt+(q-d)zt]+Y[(l-p)yz+(\gamma-q)zx$$
$$+(l-r)xy+(\alpha-d)xt+(l-d)yt+(p-d)zt]+Z[(\gamma-p)yz+(\gamma-q)zx+(\gamma-r)xy$$
$$+(\gamma-d)xt+(\gamma-d)yt+(\gamma-d)zt]=0$$

This represents a cone if

$$\left[X(s-p)+Y(b-p)+Z(c-p)\right]^{\frac{1}{2}}\left[X(a-d)+Y(r-d)+Z(q-d)\right]^{\frac{1}{2}}$$

$$\pm\left[X(a-q)+Y(s-q)+Z(c-q)\right]^{\frac{1}{2}}\left[X(r-d)+Y(b-d)+Z(p-d)\right]^{\frac{1}{2}}$$

$$\pm\left[X(a-r)+Y(b-r)+Z(s-r)\right]^{\frac{1}{2}}\left[X(q-d)+Y(p-d)+Z(c-d)\right]^{\frac{1}{2}}=0.$$

Consequently this is an equation of a Circle quartic. The four lines obtained by equating the square brackets to zero are the 8 tangents of the curve.

The tangent plane to the quadric (2) at the point $X=Y=Z=0$ on A is easily found to be

$$X\left[(a-d)x+(r-d)y+(q-d)z\right]+Y\left[(r-d)x+(b-d)y+(p-d)z\right]$$
$$+Z\left[(q-d)x+(p-d)y+(c-d)z\right]=0.$$

Now this is the polar plane of (X,Y,Z,T) with regard to the quadric cone

$$(a-d)x^2+(b-d)y^2+(c-d)z^2+2(p-d)yz+2(q-d)zx+2(r-d)xy=0$$

hence if we make the plane M the same as E it appears that the point P corresponds to the two points QQ' on A and its polar plane with regard to the cone

meets the section of its polar quadric with regard to S_3.

Let C_2 be the conic of the quadric cone and C_3 that of the cubic surface S_3 by the plane E. Then the transformation is clearly derived from C_2 and C_3 by the method of §2. The seven base points $a_1, a_2 \ldots a_7$ consequently have the same polar lines with regard to C_2 and C_3. Hence we have the following theorem.

If we project from one of the eight poles of a plane quartic to a cubic surface the other seven poles project into seven points which have the same polar lines with regard to a conic and cubic.

Taking each of the eight poles in turn we find that there are eight Aronhold systems of bitangents of a Lüroth quartic which possess the property of being the polar lines of seven points with regard to both a conic and a cubic. We are not justified in saying that every Aronhold system of bitangents possesses this property.

To obtain a set of eight associated points $A_0 \cdots A_7$ when $a_1 \cdots a_7$ are given we proceed as follows.

It is known that eight associated points are the vertices of two tetrahedra which are self polar with regard to a quadric[1]. Consequently if one of these tetrahedra be chosen as tetrahedron of reference, the coordinates of the vertices of the other can be represented by $(x_1 y_1 z_1 t_1)$, $(x_2 y_2 z_2 t_2)$ $(x_3 y_3 z_3 t_3)$, $(x_4 y_4 z_4 t_4)$ where relations of the type

$$u x_r x_s + v y_r y_s + w z_r z_s + k t_r t_s = 0, \qquad r \neq s.$$

are satisfied.

By using the known properties of an orthogonal matrix we may deduce from these six relations of the type

$$\begin{aligned}
&\alpha x_1 t_1 + \beta x_2 t_2 + \gamma x_3 t_3 + \delta x_4 t_4 = 0, \\
&\alpha y_1 t_1 + \beta y_2 t_2 + \gamma y_3 t_3 + \delta y_4 t_4 = 0, \\
&\alpha z_1 t_1 + \beta z_2 t_2 + \gamma z_3 t_3 + \delta z_4 t_4 = 0, \\
&\alpha y_1 z_1 + \beta y_2 z_2 + \gamma y_3 z_3 + \delta y_4 z_4 = 0, \\
&\alpha z_1 x_1 + \beta z_2 x_2 + \gamma z_3 x_3 + \delta z_4 x_4 = 0, \\
&\alpha x_1 y_1 + \beta x_2 y_2 + \gamma x_3 y_3 + \delta x_4 y_4 = 0.
\end{aligned}$$

Now let $t = 0$ be the plane containing the seven points

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1 Hesse, Analytische Geometrie des Raumes (vol. 1, 5th edition, Leipzig, 1896), p. 216.

... and let three of these points be at the corners of the reference tetrahedron. The coordinates of the other four may then be represented by $(x_1 y_1 z_1 0)$, $(x_2 y_2 z_2 0)$, $(x_3 y_3 z_3 0)$, $(x_4 y_4 z_4 0)$. When these are given our six equations determine the ratios of $(\alpha, \beta, \gamma, \delta)$ and of $(t_1 t_2 t_3 t_4)$ uniquely. Hence when $a_1 \ldots a_7$ are given we can calculate Frobenius' irrational invariants for the associated quartic curve and if we know the relation between $a_1 \ldots a_7$ for a particular type of quartic, we can (theoretically) find the corresponding relation between the eight associated points.

§ 5. Derivation of a (1,2) transformation by mapping the bundle of a twisted cubic on a plane.

Let the polar planes of a point P with regard to a pencil of quadrics meet in a line l. This line meets a given quadric S in two points Q, Q' which may be said to correspond to P. If moreover, P is restricted to lie in a certain plane X, there is generally only one position of P for a given position of Q, for the polar planes of Q with regard to the quadrics of the pencil intersect in a line l which meets X in P. There is consequently a $(1,2)$ transformation between the points of the plane X and the points of the quadric S.

If P moves along a straight line the corresponding line l generates a quadric surface which meets S in a biquadratic C_4. This curve C_4 passes through six fixed points $A_1 \cdots A_6$ on S, viz the points in which S is met by the twisted cubic K_3 which is the locus of the poles of X with regard to the quadrics of the pencil

If now we project the points of S on to a plane Y taking A_1 as vertex of projection, the bipuadratics C_4 project into cubics through seven fixed points viz the projections of $A_2 \cdots A_6$ and the two points where the generators through A_1 meet Y. The two points QQ' and consequently their projections on the plane Y come together when the line l touches S and then the locus of P is a quartic curve L, the limiting curve of the $(1,2)$ transformation between the two planes X and Y, or between X and S.

The 28 bitangents of L are derived from the six points $A_1 \cdots A_6$, the twelve generators through these points and the ten pairs of plane sections of S which pass through the six points. The curve on S which corresponds to L is the intersection of S with the Weddle surface which is the locus of the vertex of a quadric cone through $A_1 \cdots A_6$; for Geiser[1] has shown that this surface is also the locus of a point Q such that $A_1 \cdots A_6 QQ$ form a set of eight

1. Crelle's Journal. Bd. 67. (1867).

... to ... Now this is what happens in the present case for when Q and Q' coincide. The ∞^1 lines through P correspond to ∞^1 quadrics through $A_1 \ldots A_6\, Q\, Q$ and since S also passes through these points it follows that the eight points form an associated set.

To find the equation of L we shall suppose that the pencil of quadrics is given by the equation

$$(a+\lambda)x^2 + (b+\lambda)y^2 + (c+\lambda)z^2 + (d+\lambda)t^2 = 0,$$

where λ is a variable parameter.

If (X, Y, Z, T) are the coordinates of P the line common to the polar planes of P has coordinates proportional to

$$(b-c)YZ,\ (c-a)ZX,\ (a-b)XY,\ (a-d)XT,\ (b-d)YT,\ (c-d)ZT,$$

and generates a tetrahedral complex.

The condition that this line should touch the quadric S is of the fourth degree in X, Y, Z, T. If in particular, the equation of S is

$$lx^2 + my^2 + nz^2 + pt^2 = 0,$$

the ... dition is

$$bcl(b-c)^2Y^2Z^2 + amp(c-a)^2Z^2X^2 + abp(a-b)^2X^2Y^2 + amn(a-d)^2X^2T^2 + bnl(b-d)^2Y^2T^2 + lmn(c-d)^2Z^2T^2 = 0$$

$$l = (a-b)(a-c)(a-d), \qquad m = -(b-a)(b-c)(b-d),$$

$$n = (c-a)(c-b)(c-d), \qquad p = (d-a)(d-b)(d-c),$$

the equation becomes

$$\lambda\mu\nu\rho(\lambda+\mu+\nu+\rho)^2 + \text{cyclic} = 0$$

and represents a desmic quartic surface. The curve L being a plane section of this surface is a desmic quartic curve.

If in the general case the equation of the plane X is

$$\xi x + \eta y + \zeta z + \tau t = 0,$$

the equations of the conic K_3 are

$$x : y : z : t = \frac{\xi}{a+\lambda} : \frac{\eta}{b+\lambda} : \frac{\zeta}{c+\lambda} : \frac{\tau}{d+\lambda}$$

where λ is a variable parameter.

Taking two points on the cubic with parameters λ and μ we see that the coordinates of the line joining them are proportional to six quantities of the type

$$p_{23} = \frac{(d-a)\,\xi\tau}{(a+\lambda)(a+\mu)(b+\lambda)(b+\mu)}$$

Comparing these with the coordinates of the line l, we see that this chord of K_3 is the line l corresponding to a point P

...ll coordinates

$$X = \frac{\rho\,\dots\,\tau\,\sigma}{\delta(a-b)(a-c)(a-d)} \qquad Y = \frac{\dots\,\sigma\,\dots}{\eta(b-c)(b-c)(b-d)}$$

$$Z = \frac{(c+\lambda)(c+\mu)}{\dots(c-\dots)(c-b)(c-d)} \qquad T = \frac{(d+\lambda)(d+\mu)}{\dots(d-a)(d-b)(d-c)}$$

and the ... - line in the plane X. Hence a point ... in the plane X corresponds to a chord of the twisted cubic K_3. A ... t of the twisted cubic corresponds to a point P_0 of a certain conic C_2 obtained by putting $\lambda = \mu$ in the above equations. If a point P moves along a tangent to C_2 the corresp... chord l always ... through a fixed point on K_3. Hence a point on K_3 is associated with a tangent to C_2, the chord joining two points on K_3 with the point of intersection of ... two associated tangents.

It is now evident that the six bitangents of L which correspond to the points $A_1\ A_2 \dots A_6$ are tangents to the conic C_2. They ... form a Brianchon set.

It should be noticed that the polar planes of a point K_3 with regard to the pencil of quadrics meet in ... line

which lies in the plane X and touches the conic C_2. This A_2 corresponds to a line and not to a single point

An interesting theorem may be obtained by considering the case when the chords joining five points on K_3 all belong to a quadratic complex R_2. It is clear from the expressions for the coordinates of a line l that the points in the plane X corresponding to these chords all lie on a quartic curve C_4. Now these points are the ten points of intersection of five tangents to C_2 hence it follows that the curve C_4 is a Lüroth quartic. Making use of the fact that there are ∞^1 pentagons circumscribed to C_2 and inscribed to C_4 we obtain the following theorem.

If a quadratic complex R_2 and a twisted cubic K_3 are such that a set of five points can be found on K_3 whose joining lines all belong to R_2, then ∞^1 such sets of five points on K_3 can be found

There is a similar theorem for a complex of degree n and sets of $2n+1$ points on a twisted cubic.

It should be remarked that the quadratic complex R_2 cannot degenerate into the complex of tangents to a quadric surface

for a quadric surface cannot touch the ten lines joining five distinct points. Everything points to the conclusion that the quadratic complex is of a special type but the relation between its invariants has not yet been found. It is clear that if one set of five distinct points can be found whose joins belong to the complex there are at least ∞^3 others for we may take any one of the ∞^2 twisted cubics on the five points and there will be ∞^1 sets of five points situated thereon. In accordance with the general phenomena of poriems we should expect there to be ∞^6 sets of five points which satisfy the requirement.

If we reciprocate with regard to one of the fundamental linear complexes of R_2 we find that when sets of five points exist whose joins belong to R_2 there are also sets of five planes whose lines of intersection belong to R_2 and the sets of points and planes are equally numerous.

It is evident from what has gone before that if we reciprocate with regard to one of the quadrics of our pencil, the points on the Lüroth quartic C_4 correspond to the planes [illegible] lines [illegible]

contacts of quads on F_3. Now these must generate a cubic surface R_3 having K_3 as a four fold curve and our theorem tells us that the tangent cone to R_3 from any point of F_3 is the reciprocal of the Steinerth quartic C_4 with regard to some quadric of the pencil.

It can also be shown that any plane section of R_3 can be derived from C_4 by a suitable quadratic transformation.

Let Y be the plane of the section, then the transformation under consideration is obtained by finding where the line l corresponding to a point P in X. meets Y. Now if S. S' are two quadrics of the pencil there is a correlation between P and the line p in which its polar plane with regard to S meets Y. similarly there is a correlation between P and the line p' in which its polar plane with regard to S' meets Y. If p and p' meet in Q there is a quadratic transformation[1] which sends P into Q The base points in the plane Y are the points where this plane meets K_3.

We shall now make further use of this quadratic transformation.

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[1] T. Reye. Zeitschr. für Math. u Phys. Bd. 11. Geometrie der Lage 3rd ed. Part 2. §25.

Let $A_1 A_2 \ldots A_6$ be six arbitrary points on K_3 : the lines joining them correspond in X to the points of intersection of six tangents to the conic C_2 when they meet the plane Y in fifteen points lying on the section of the Weddle surface having $A_1 \ldots A_6$ as nodes. This section is a quartic curve which also passes through the three base points in which K_3 meets Y. The quadratic transformation consequently sends it into a quintic curve C_5 with three nodes and this quintic passes through the fifteen intersections of the six tangents to C_2. Now Darboux has shown that when a quintic C_5 and a conic C_2 are such that a set of six tangents to C_2 intersect in fifteen points lying on C_5 there are ∞' such sets.

Transforming this theorem we are led to the conclusion that there are ∞' sets of points $B_1 \ldots B_6$ on K_3 whose joining lines cut out a configuration of 15 points on a plane section of the Weddle surface having $A_1 \ldots A_6$ as nodes. This result has been proved by some other method[1]

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1. H. Bateman, Proc. London. Math. Soc. Ser. 2. Vol. 3. (1905) p. 288

Morley and Conner, Amer. Journal, Vol 31. p. 263.

It should be noticed that the correspondence between a point P of the plane X and a chord l of the twisted cubic K_3 enables us to map the Weddle surface W on a plane[1] by making P correspond to the pair of points in which l meets W. The ∞^3 plane sections of W are then represented by trinodal quintics which pass through the fifteen intersections of six tangents to the conic C_2.

The result that there are ∞^3 trinodal quintics through the fifteen points is unexpected as the method of counting constants suggests only ∞^2. The result may, however, be verified as follows.

Let $x:y:z = 1:\theta:\theta^2$ be the parametric representation of the conic C_2 and if $\alpha_1, \alpha_2 \ldots \alpha_6$ are the parameters of the points of contact of the six tangents and $\beta_1, \beta_2, \beta_3$ are three arbitrary parameters the equation

$$\frac{\prod_1^6 (\theta - \alpha_s)}{\quad} = \frac{\prod_1^6 (\phi - \alpha_s)}{\quad} ;$$

in Darboux coordinates (θ, ϕ) represents a quintic curve[2]

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1. Various parametric representations of W are given in Baker's Multiply Periodic Functions

2. Cf. Darboux. Sur une classe remarquable de courbes et de surfaces algébriques

... passes through the fifteen intersections of the tangents to C_2 at the points x_1 x_2 and has double points at the points of intersection of the tangents at the points $(x_1 . x_2 x_3)$. (Cf. A.C. Dixon Quarterly Journal. Vol 26 p 212. 1893)

The theorem can ... be generalised as follows.

There are n^n curves of degree $2n$ which pass through the $n(n+1)$ intersections of n tangents to a conic and have double points at the $\frac{1}{2}n(n-1)$ intersections of n tangents to the conic.

"Let us next consider a set of five points $B_1 B_2 \cdots B_5$ on the curve K_3. If l is any chord of K_3. the twisted cubics that pass through $B_1 B_2 . B_5$ and meet l generate a surface of the fifth degree S_5 having K_3 as double curve. This surface has been discussed by Clebsch[1] who shows that it can be represented on a plane in such a way that the images of the plane sections are Lüroth quartics passing through a fixed point T' and the ten intersections of five li...

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1. Math. Ann. Bd 1. 1869. p 253. Bd 4. p 249. See also. Sturm. Geometrischen Verwandtschaften. Bd. 4. p 315.

The surface S_5 meets a plane Y in a quintic with double points at the three base points in which K_3 meets Y. This quintic Γ_5 contains a configuration of 10 points which lie by threes on ten lines, viz the points where the joins of $B_1 \ldots B_5$ meet Y (for since $B_1 \ldots B_5$ are triple points of S_5 the lines joining them

The quintic Γ_5 is transformed by any quadratic transformation into a cubic curve Γ_3 passing through the intersections of 5 tangents[1] to the conic C_2. Now since there are ∞^1 configurations of 5 points on C_2 which lie by fours on five tangents to C_2 it follows that there are likewise ∞^1 configurations of the points on Γ_5 lying by threes on ten lines.

Hence any plane section of S_5 contains ∞^1 configurations of ten points lying by threes on ten lines.

If the plane Y passes through the line l, the curve Γ_5 breaks up into the line l and a quartic with a double point.

1. The fact that the ten points in which the joins of five points in space and an arbitrary plane can be transformed by a quadratic transformation into the ten

Hence we are led to the conclusion that a quartic with a double point can sometimes contain ∞^1 configurations of ten points lying by threes on ten lines. The corresponding Sürath quartic has a double point

The skew cubic which generates S_5 cuts out a series of triads of points on Γ_5 which together with the three base points cut out by K_3 are all conjugate triads[1] with regard to a certain conic Γ_2. Hence any such triad of points lies on a conic through the three base points and so transforms into a set of three collinear points on C_4. These points are moreover associated with a fixed point on C_4, viz the point corresponding to the line l. To see this we have only to remark that any generating cubic of S_5 lies on a quadric through K_3 and this quadric contains the line l since it is met by it in three points. The conic through the base points which contains the

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1 [illegible]

A. C. Dixon. Quarterly Journal. Vol 23. 1889.

G. Humbert. Journal de l'École Polytechnique. Cah. 64 (1894).

tuned of points cut out by this generating cubic consequently passes through the point in which l intersects Y and so transforms into a line passing through a fixed point T' of the Lüroth quartic C_4.

When the generating cubic touches the plane Y, the associated line will touch the Lüroth quartic. Hence the ten tangents from T' to C_4 are derived from the ten generating cubics which touch the plane Y. Now it is clear that the ten points of contact of these cubics lie on the conic Γ_2 and are in fact the ten points in which Γ meets Γ_5. Transforming this result we find that the points of contact of the ten tangents from T to C_4 lie on a quartic curve with three biflecnodes which lie on C_4.

It follows from the plane representation of the surface S_5 that the point T can be regarded as an arbitrary point on the Lüroth quartic C_4, hence we have the following theorem.

If tangents are drawn to the curve from an arbitrary point T on a Lüroth quartic, the ten points of contact lie on a quartic curve with three biflecnodes situated on the Lüroth quartic. The lines joining these nodes touch the curve and form with the two tangents from T to C_2 a set of

five lines intersecting in ten points on the Lüroth quartic.

It is clear that the three bifid cuspidal [illegible] when T_1, T_2 and T_3 are [illegible]

The second part of the theorem follows at once when we consider the set of 5 points on K_3 consisting of the two points in which it is met by l and the three points in which it is met by Y

1. Generation of a quartic curve from a quadratic transformation between the lines of a plane

F. Godt[1] and E. Timerding[2] have shown that the general quartic curve can be derived from a quadratic transformation between the lines of a plane by considering the locus of points which lie on their [illegible][3]

Let ABC, $A'B'C'$ be the two sets of fundamental points of the transformation, (ξ, η, ζ), (ξ', η', ζ') the coordinates of a line referred to these triangles, then a point whose equation is

$$\xi x + \eta y + \zeta z = 0,$$

corresponds to a conic whose tangential equation is

$$\frac{x}{\xi'} + \frac{y}{\eta'} + \frac{z}{\zeta'} = 0.$$

The point equation of this conic is

$$\sqrt{x\xi'} + \sqrt{y\eta'} + \sqrt{z\zeta'} = 0.$$

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1. Dissertation Göttingen (1893), Clebsch. Lindemann, Vorlesungen. p. 1007.

2. Math. Ann. Bd. 53. (1900). p. 193.

3. This method of generation and a number of others are included in a general scheme studied by Caporali. Memorie di Geometria (1888) and Segre. Annali di Mat. (2) 20 (1892)

Let $x' = a_1x + b_1y + c_1z$, $y' = a_2x + b_2y + c_2z$, $z' = a_3x + b_3y + c_3z$

be the relations connecting the two systems of point coordinates. Then the point (x,y,z) lies on its corresponding conic if

$$\sqrt{x(a_1x+b_1y+c_1z)} + \sqrt{y(a_2x+b_2y+c_2z)} + \sqrt{z(a_3x+b_3y+c_3z)} = 0$$

Comparing this equation with the equations obtained in sections 3 and 4 for the quartics of Lüroth and Humbert we can easily find the equations of quadratic transformations which find its these quartics.

9 4. Kubic quartic considered as the limiting curve of a (1,2) transformation

The general equation of a cubic through the seven points P_ν with [illegible]

$$x_\nu = \lambda^{(\nu)} \qquad y_\nu = \lambda^{\nu} \qquad z_\nu = \lambda^{\nu} \qquad (\nu = 1)$$

where $\epsilon^{\nu} = 1$ [illegible]

$$\xi(x^3 - yz^2) + \eta(y^3 - zx^2) + \zeta(z^3 - xy^2) = 0 .$$

Consider the transformation

$$\frac{X}{x^3 - yz^2} = \frac{Y}{y^3 - zx^2} = \frac{Z}{z^3 - xy^2} \qquad (1)$$

which makes the line

$$\xi X + \eta Y + \zeta Z = 0$$

correspond to a cubic curve through the seven points P_ν:

It is evidently a (1,2) transformation whose Jacobian is the quartic curve

$$J = \begin{vmatrix} 3x^2 & -z^2 & -2yz \\ -2xz & 3y^2 & -x^2 \\ -y^2 & -2xy & 3z^2 \end{vmatrix} = 0$$

or $J = 27x^2y^2z^2 + x^3y^3 + y^3z^3 + z^3x^3 - 3x^2y^2z^2 = 0$

To find the limiting curve of the transformation we must eliminate (x,y,z) from the first equation and the equations (1)

We easily find that

$$Xz + Yx + Zy = 0 . \qquad (2)$$

and $J=0$ is equivalent to

$$X x^2 y + Y y^2 z + Z z^2 x = 0 .$$

consequently

$$\frac{X}{x(x^3 - y^3)} = \frac{Y}{x(z y^2 - z^2)} = \frac{Z}{y(y z^2 - x^3)}$$

or
$$\frac{X}{xY} = \frac{Y}{xZ} = \frac{Z}{yX} .$$

Hence
$$\frac{X^2}{Y} + \frac{Y^2}{Z} + \frac{Z^2}{X} = 0$$

or
$$X^3 Z + Y^3 X + Z^3 Y = 0$$

the equation of [illegible] curve [illegible]

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1. Klein-Fricke. Vorlesungen über die Theorie der elliptischen Modulfunktionen Leipzig (1892). pp. 675. 678. 701 . Math. Ann. Bd. 14. (1879). p 428

Haskell. American Journal Vol 13 (1890). H.F. Baker. Multiply Periodic Functions. p 604

is a Klein's quartic.

The line joining the two points (x,y,z) corresponding to a given point (X,Y,Z) is represented by the equation

$$Xx + Yx + Zy = 0.$$

There is evidently a correlation connecting this line and the point (X,Y,Z) which will be denoted hereafter by C. If Q lies on L the two corresponding points P are consecutive, and the line joining them envelops the curve whose tangential equation is

$$\xi^3\xi + \xi^3\eta + \eta^3\zeta = 0,$$

If PP' are the two consecutive points, P lies on the curve J and the line PP' may be called the principal direction at P. A curve which crosses J at P in a direction different from the principal one corresponds to a curve which touches the limiting curve L at Q. For instance, a line which meets J in six distinct points generally corresponds to a rational cubic touching L in six distinct points.

To find the equation of this cubic, we take the equation

of the line in the form

$$\xi x + \eta y + \zeta z = 0. \qquad (3)$$

The equation can be put in the form

$$\frac{x}{\eta X - \zeta Z} = \frac{y}{\zeta Y - \xi X} = \frac{z}{\xi Z - \eta Y}.$$

Now $$X y^2 + Y z^2 + Z x^2 = 0,$$

$$\xi(\eta X - \zeta Z)^2 + \eta(\zeta Y - \xi X)^2 + \zeta(\xi Z - \eta Y)^2 = 0$$

is the equation of this cubic. The coordinates of the double points are evidently

$$X = \zeta^2, \quad Y = \xi^2, \quad Z = \eta^2;$$

the point (ζ^2, ξ^2, η^2) is the intersection of the lines $(1, 1, 1)$ and the conic mentioned.

A line (3) which passes through one of the base points P_ν corresponds to a rational cubic which breaks into factors. One of the factors is the line

$$\epsilon^{4\nu} Y + \epsilon^{\nu} Z + \epsilon^{2\nu} X = 0 \qquad (4)$$

associated with the base point P_ν. This line is a bitangent of L. The other factor is a conic

$$\epsilon^{5\nu}\xi^2 X^2 + \epsilon^{3\nu}\eta^2 Y^2 + \epsilon^{6\nu}\zeta^2 Z^2 - (\eta^2 + 2\xi\eta\epsilon^{3\nu}) YZ - (\zeta^2 - 2\eta\zeta\epsilon^{4\nu}) ZX - (\xi^2 + 2\xi\zeta\epsilon^{2\nu}) XY = 0.$$

which touches the quartic L in four points. If the line passes through a second base point P_μ the conic breaks up into a [illegible]

$$\epsilon^{4\mu} Y + \epsilon^{\mu} Z + \epsilon^{2\mu} X = 0$$

corresponding to P_μ and a second bitangent

$$\epsilon^{4\nu-2\mu} \xi^2 X + \epsilon^{2\mu-4\mu} \eta^2 Y + \epsilon^{6\nu-\mu} \zeta^2 Z = 0.$$

corresponding to the line $P_\lambda P_\nu$.

Since [illegible]

$$\xi \epsilon^{4\mu} + \eta \epsilon^{\mu} + \zeta \epsilon^{2\mu} = 0,$$

the equation of the [illegible] written in the form

$$(\epsilon^{\mu} - \epsilon^{\nu})^2 X + (\epsilon^{2\mu} - \epsilon^{2\nu})^2 Y + \epsilon^{\mu}\epsilon^{\nu} (\epsilon^{3\mu} - \epsilon^{3\nu})^2 Z = 0. \quad (5)$$

The quartic curve L has seven bitangents with equations of type (4) and twenty one with equations of type (5). Other equations for the bitangents and quadritangent conics have been obtained by Klein. It should be mentioned that many writers have used a different system of coordinates in studying the properties of the curve[1].

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[1] See for instance Weber, Acta Holm. Bihang. till Handlingar. 21 (1895) N:3. A. B. Coble, American Journal Vol. 28 (1906).

A set of eight associated points in space from which Klein's quartic may be derived by Hesse's method, is easily deduced from our set of seven points. If we take four of the points at the corners of the tetrahedron of reference the other four points may be represented by

$(1, -\theta, 1+\theta, 2\theta+1)$, $(1+\theta, 1, 1, -1)$, $(-\theta, -1, \theta, \theta+1)$, $(\theta, 1+\theta, 1, \theta)$

where $\theta = \epsilon^3 + \epsilon^5 + \epsilon^6$

and consequently $\theta^2 + \theta + 2 = 0$.

8. Reduction of the equation of a general quartic to a special form.

Loiman[1] has shown that the general quartic ... the polars ... of the points of ... with regard to a cubic C_3 ... the envelope of the ... of points on C_2 with regard to C_3.

Let ... be the parametric equations of C_2, then the equation of the polar conic with regard to C_3 of a point on C_2 with ... is

$$A \equiv \theta^2 S_0 + 2\theta S_1 + S_2 = 0 .$$

and the envelope of the conic is the quartic curve

$$Q \equiv S_0 S_2 - S_1^2 = 0 .$$

If we use Darboux coordinates (θ, ϕ) which are such that

$$x = \theta\phi , \quad y = \theta + \phi , \quad z = 1 .$$

are the coordinates of an arbitrary point T, and θ, ϕ the

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1. Higher Plane Curves. See also Gerbaldi, Rend. Palermo, 1892, Ciani, Rend. Lombardo, 1903.

parameters of the points of contact of the two tangents from P to C_2. it follows that the polar conic of P. viz

$$N \equiv \theta\phi\, S_0 + (\theta+\phi)\, S_1 + S_2$$

passes through the poles of the two tangents through P to C_2 with regard to C_3 and therefore through the points of contact of the conics

$$A \equiv \theta^2 S_0 + 2\theta\, S_1 + S_2 = 0,$$

$$B \equiv \phi^2 S_0 + 2\phi\, S_1 + S_2 = 0,$$

with their envelope Q.

The conic N consists of two straight lines if P lies on the Hessian H of C_3 and then the two lines intersect in the corresponding point P' on H [illegible] The condition that this is that $H(x,y,z) = 0$, or

$$H[\theta\phi,\ \theta+\phi,\ 1] = 0.$$

where H is homogeneous and of the third degree.

We are now led to ask the following question:

[illegible]

$A \equiv \theta^2 S_0 + 2\theta S_1 + S_2 \ldots$,

$\ldots + 2\phi S_1 + S_2 \ldots$

$\ldots + \ldots S_1 + S_2 \ldots$

such that the points of contact with Q of each of the three pairs BC, CA, AB lie on two straight lines.

The question is evidently equivalent to the following.

Can we find a triangle which is inscribed in H and circumscribed to Q_2?

The answer seems to be that we can generally find two proper triangles of the required type. For if we eliminate ψ from the equations

$$H[\theta\psi, \theta+\psi, 1] = 0, \qquad H[\phi\psi, \phi+\psi, 1] = 0$$

by means of a determinant using Sylvester's method, we are led to an equation of the ninth degree in θ and of the ninth degree in ϕ. The first three rows of the determinant contain only θ, the last three rows contain

$\ldots \phi \ldots$

the determinant. Rejecting this factor we are left with an equation of the sixth degree in $\theta\phi$, $\theta+\phi$, 1, which

[illegible] in the form

$$F(x, y, z) = 0$$

represents a curve of the sixth degree.

The curves $F = 0$, $H = 0$ intersect in 12 points but these do not all give rise to proper triangles. If RS is a common tangent to F and [illegible] R being on F, and T is a point in which the other tangent from R to C_2 meets C_3, RRT must be regarded as a degenerate triangle fulfilling the conditions, and so R is one of the points of intersection of F and H. [illegible] C_2 and H generally have twelve common tangents there are generally twelve points of type R.

The points of intersection of C_2 and H do not give [illegible] Clifford [illegible] Tucker has only [illegible]

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1. [illegible] Math Papers. p. 17

Clifford makes a few remarks on the present problem but arrives at a slightly different result

twelve intersections by means of degenerate cases we must conclude that there are six other intersections which give rise to two proper triangles fulfilling the conditions. This number agrees with the well known result for the case when Γ_2 and H consists of three straight lines[1]. Our argument is not quite conclusive. It would be more satisfactory if the algebraic work could be carried through in detail.

There is an important exceptional case in which ∞^1 complete quadrilaterals can be inscribed in H and circumscribed to Γ_2. Darboux[2] and Clifford[3] have shown in fact that if one complete quadrilateral is circumscribed to Γ_2 and completely inscribed in H, then ∞^1 such quadrilaterals can be found. This is to be expected because a quadrilateral inscribed in H and circumscribed to Γ_2 gives four triangles fulfilling the conditions and this is more than the proper number consequently there must be an infinity of triangles.

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1. Salmon's Conic Sections, p. 273. The reciprocal theorem is given.

2. Loc. Cit. 3. Math. Papers p. 29 205.

It will be convenient to call this exceptional case the case D and to refer to the special type of quartic which arises from it as a "D" quartic. It will be shown presently that this quartic is a desmic quartic.

In the general case the quartic can be put into the form where is taken to be a section of the quartic curve by the form

$$\frac{l}{x_1x_2} + \frac{m}{y_1y_2} + \frac{n}{z_1z_2} = 0$$

Let us write

$$x_1x_2 \equiv \phi\psi\, S_0 + (\phi+\psi)\, S_1 + S_2 ,$$

$$y_1y_2 \equiv \psi\theta\, S_0 + (\psi+\theta)\, S_1 + S_2 ,$$

$$z_1z_2 \equiv \theta\phi\, S_0 + (\theta+\phi)\, S_1 + S_2 .$$

We can then find constants α, β, γ such that

$$\alpha y_1y_2z_1z_2 + \beta z_1z_2x_1x_2 + \gamma x_1x_2y_1y_2 \equiv S_0S_2 - S_1^2 .$$

To see this let us put

$$S_0 = 1 , \quad S_1 = t , \quad S_2 = t^2 ,$$

then the above identity holds if

$$\alpha(t+\theta)^2(t+\phi)(t+\psi) + \beta(t+\phi)^2(t+\psi)(t+\theta) + \gamma(t+\psi)^2(t+\theta)(t+\phi) \equiv 0 .$$

Now the equation can be written in the form

$$\frac{x}{\phi-\psi} + \frac{y}{\psi-\theta} + \frac{z}{\theta-\phi} = 0$$

and then it is easy to verify that we have identically

$$(\phi-\psi)y_1y_2z_1z_2 + (\psi-\theta)x_1x_2z_1z_2 + (\theta-\phi)x_1x_2y_1y_2 \equiv (\phi-\psi)(\psi-\theta)(\theta-\phi)(S_0S_2-S_1^2)$$

Thus the equation of the second quartic curve may be expressed in the form

$$(\phi-\psi)y_1y_2z_1z_2 + (\psi-\theta)x_1x_2z_1z_2 + (\theta-\phi)x_1x_2y_1y_2 = 0,$$

If we change the arbitrary constants in $x_2y_2z_2$ the equation may be written

$$Y_1Y_2Z_1Z_2 + Z_1Z_2X_1X_2 + X_1X_2Y_1Y_2 = 0$$

It is clear that the quartic curve passes through the four points of intersection of two pairs of lines such as $X_1X_2 = 0$ $Y_1Y_2 = 0$ and that the three pairs of lines give rise to twelve points as in the figure.

The conics A, B, C have the respective equations

$$Y_1Y_2 + Z_1Z_2 = 0 , \quad Z_1Z_2 + X_1X_2 = 0 , \quad X_1X_2 + Y_1Y_2 = 0 .$$

It is easy to verify in fact that each of these conics touches X at four points.

It should be noticed that the four points in which A cuts $X_1X_2 = 0$, the four points in which B cuts $Y_1Y_2 = 0$ and the four points in which C cuts $Z_1Z_2 = 0$, all lie on the conic

$$X_1X_2 + Y_1Y_2 + Z_1Z_2 = 0$$

This may be written in the alternative forms

$$\frac{\alpha_1\alpha_2}{\lambda - \mu} + \frac{\beta_1\beta_2}{\mu - \nu} + \frac{\gamma_1\gamma_2}{\nu - \lambda} = 0$$

$$(\sigma_2^2 - 3\sigma_3\sigma_1)\,S_0 - 2\sigma_3\,S_1 + (\sigma_1^2 - 3\sigma_2)\,S_2 = 0 .$$

where

$$\sigma_1 = \lambda + \mu + \nu , \qquad \sigma_2 = \mu\nu + \nu\lambda + \lambda\mu , \qquad \sigma_3 = \lambda\mu\nu .$$

9.4 Configurations of order four which have a quartic curve of type D

If in the case D θ, ϕ, ψ, χ are the parameters of the points of contact with C_2 of the sides of one of the quadrilaterals, we have two pairs of tangent lines

$$x_1x_2 \equiv \phi\psi\, S_0 + (\phi+\psi)S_1 + S_2 \,, \quad u_1u_2 \equiv \theta\chi S_0 + (\theta+\chi)S_1 + S_2 \,,$$

$$y_1y_2 \equiv \psi\theta\, S_0 + (\psi+\theta)S_1 + S_2 \,, \quad v_1v_2 \equiv \phi\chi S_0 + (\phi+\chi)S_1 + S_2 \,,$$

$$z_1z_2 \equiv \theta\phi\, S_0 + (\theta+\phi)S_1 + S_2 \,, \quad w_1w_2 \equiv \psi\chi S_0 + (\psi+\chi)S_1 + S_2 \,.$$

The equation of the quartic curve may be thrown into the forms

$$(\phi-\psi)\, y_1y_2z_1z_2 + (\psi-\theta)\, z_1z_2x_1x_2 + (\theta-\phi)\, x_1x_2y_1y_2 = 0 \,,$$

$$(\phi-\psi)\, v_1v_2w_1w_2 + (\psi-\chi)\, w_1w_2z_1z_2 + (\chi-\phi)\, z_1z_2v_1v_2 = 0 \,,$$

$$(\psi-\theta)\, w_1w_2u_1u_2 + (\theta-\chi)\, u_1u_2y_1y_2 + (\chi-\psi)\, y_1y_2w_1w_2 = 0 \,,$$

$$(\theta-\psi)\, u_1u_2v_1v_2 + (\phi-\chi)\, v_1v_2z_1z_2 + (\chi-\theta)\, z_1z_2u_1u_2 = 0 \,,$$

or in the form

$$\lambda\, x_1x_2u_1u_2 + \mu\, y_1y_2v_1v_2 + \nu\, z_1z_2w_1w_2 = 0 \,,$$

where λ, μ, ν are subject to the relation $\lambda+\mu+\nu=0$ but are otherwise arbitrary. It would be noted that

i) $(\phi-\psi)(\theta-\chi)\, x_1x_2u_1u_2 + (\psi-\theta)(\phi-\chi)\, y_1y_2v_1v_2 + (\theta-\phi)(\psi-\chi)\, z_1z_2w_1w_2 \equiv 0$

It is clear that the quartic curve passes through the four vertices of points in which the six pairs of lines u_1u_2, v_1v_2, w_1w_2, x_1x_2, y_1y_2, z_1z_2 intersect. From what has gone before it appears that the three diagonal points of each of these tetrads lie on the Hessian H of the cubic curve C_3, while the six pairs of lines touch the Cayleyan Γ_3 of the cubic.

If through the vertices U_1 of the quadrilateral

$$(\phi-\psi)\,u_1u_2 + (\psi-\theta)\,v_1v_2 + (\theta-\phi)\,w_1w_2 = 0\,,$$

$$(\phi-\psi)\,u_1u_2 - (\phi-\chi)\,y_1y_2 + (\psi-\chi)\,z_1z_2 = 0\,,$$

$$(\psi-\theta)\,v_1v_2 - (\psi-\chi)\,z_1z_2 + (\theta-\chi)\,x_1x_2 = 0\,,$$

$$-(\phi-\theta)\,w_1w_2 - (\theta-\chi)\,x_1x_2 + (\phi-\chi)\,y_1y_2 = 0$$

Consequently we may write

$$(\psi-\theta)\,v_1v_2 + (\phi-\chi)\,y_1y_2 \equiv (\psi-\chi)\,z_1z_2 - (\theta-\phi)\,w_1w_2 \quad \ldots\ L$$

$$(\phi-\psi)\,u_1u_2 + (\theta-\chi)\,x_1x_2 \equiv (\phi-\chi)\,y_1y_2 - (\psi-\theta)\,v_1v_2 \quad \ldots\ M$$

$$(\theta-\phi)\,w_1w_2 + (\psi-\chi)\,z_1z_2 \equiv (\theta-\chi)\,x_1x_2 - (\theta-\psi)\,u_1u_2 = N\,.$$

and the equation of the quartic curve takes the form

$$\frac{\lambda}{(\theta-\phi)(\psi-\chi)}(M^2-N^2) + \frac{\mu}{(\psi-\theta)(\phi-\chi)}(N^2-L^2) + \frac{\nu}{(\phi-\theta)(\psi-\chi)}(L^2-M^2) = 0 \qquad (2)$$

or

$$\sigma L^2 + \rho M^2 + \tau N^2 = 0\,.$$

where $\sigma + \rho + \tau = 0$.

The conics L, M, N are so related that the equations

$$M \pm N = 0, \quad N \pm L = 0, \quad L \pm M = 0,$$

all represent pairs of straight lines. their equations must consequently be of the form

$$L \equiv BC + AD = 0, \quad M \equiv CA + BD = 0, \quad N \equiv AB + CD = 0,$$

where A, B, C, D are linear functions of the coordinates.

The equation of the quartic curve may now be written in the form

$$\sigma(B^2C^2 + A^2D^2) + \rho(C^2A^2 + B^2D^2) + \tau(A^2B^2 + C^2D^2) = 0,$$

which shows that it is a desmic quartic, i.e. a plane section of the desmic surface studied by Stephanos[1] Humbert[2] Schroeter[3] and others.

It should be noticed that an equation of the form

$$p x_1 x_2 u_1 u_2 + q y_1 y_2 v_1 v_2 + r z_1 z_2 w_1 w_2 = 0,$$

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1. Darboux's Bulletin (1874). sér. 2. t. 3. p.

2. Liouville's Journal sér. 4. t. 7 p. 353 (1891)

3. Crelle's Journal Bd. 104 p. 341. (1893).

always represent a desmic quartic, for with the aid of the identical relation (1) we may reduce it to the form (2) we may also reduce it to the form

$$F x_1 x_2 u_1 u_2 = G y_1 y_2 v_1 v_2 .$$

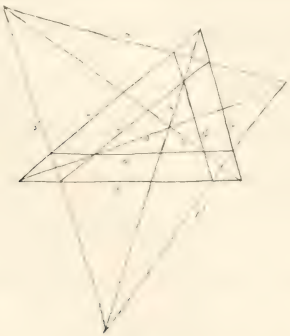
which indicates that the curve passes through the sixteen points of the configuration. Hence there are ∞^1 desmic quartics through these sixteen points

§ 10 Discussion of the case D with the aid of elliptic functions.[1]

Let the tangents of the Cayleyan Γ_3 be given parametrically by the equations

$$\xi = \wp'(\sigma) \quad , \quad \eta = \wp(\sigma) \quad , \quad \zeta = 1 \, ;$$

and let aa', bb', cc', $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ be the parameters of the pairs of lines xx', yy', zz', uu', vv', ww' respectively. Using the following figure to determine the incidences.



it is easy to see that we have the following relations.

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1 The developments of this section are nearly all due to Humbert. Loc cit. I had rediscovered them before I had proved that the D quartic is a desmic quar

$$\left.\begin{array}{llll} \alpha+b+c \equiv 0 & \beta+c+a \equiv 0 & \gamma+a+b \equiv 0 & \alpha+\beta+\gamma \equiv 0 \\ \alpha+b'+c' \equiv 0 & \beta+c'+a' \equiv 0 & \gamma+a'+b' \equiv 0 & \alpha'+\beta'+\gamma \equiv 0 \\ \alpha'+b+c' \equiv 0 & \beta'+c+a' \equiv 0 & \gamma'+a+b' \equiv 0 & \beta'+\gamma'+\alpha \equiv 0 \\ \alpha'+b'+c \equiv 0 & \beta'+c'+a \equiv 0 & \gamma'+a'+b \equiv 0 & \gamma'+\alpha'+\beta \equiv 0 \end{array}\right\} \text{ mod } 2\omega, 2\omega'$$

These give
$$\left.\begin{array}{lll} 2\alpha'-2\alpha \equiv 0 & 2\beta'-2\beta \equiv 0 & 2\gamma'-2\gamma \equiv 0 \\ 2a'-2a \equiv 0 & 2b'-2b \equiv 0 & 2c'-2c \equiv 0 \end{array}\right\} \text{ mod } 2\omega, 2\omega'$$

$$2(a+b+c) \equiv 0$$

We may satisfy the conditions by writing

$$\left.\begin{array}{lll} a' = a+\omega & b' = b+\omega & c' = c+\omega \\ \alpha \equiv a+\omega' & \beta \equiv b+\omega' & \gamma \equiv c+\omega' \\ \alpha' \equiv a+\omega+\omega' & \beta' \equiv b+\omega+\omega' & \gamma' \equiv c+\omega+\omega' \end{array}\right\} \quad a+b+c \equiv \omega' \quad \left\} \text{ mod } 2\omega, 2\omega'\right.$$

where $2\omega, 2\omega'$ are the periods of the Weierstrassian elliptic function $\wp(u)$.

Now these equations indicate that the points of contact of the four tangents $(a\ a'\ \alpha\ \alpha')$ to Γ_3 lie on a fourth tangent to Γ_3, viz. $(2a)$. Similarly for the tangents (b, b', β, β') and (c, c', γ, γ'). Now since $2(a+b+c) \equiv 0$ the tangents $(2a), (2b), (2c)$ appear that the configuration of points and lines is identical with one whose reciprocal [illegible] ... Hesse.[1]

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1 Crelle's Journal. Bd. 36. (1842). p. 153.

If the coordinates x_0, y_0, z_0 of the point Γ are such that the parameters $2a, 2b, 2c$ satisfy the equation

$$x_0 \wp'(\tau) + y_0 \wp(\tau) + z_0 = 0$$

To find the equations of the four tangents at the points where $(2a)$ meets Γ_3 again, we observe that there must be a relation of the form[1]

$$\left[x \wp'(a) + y \wp(a) + z\right]\left[x \wp'(a+\omega) + y \wp(a+\omega) + z\right]\left[x \wp'(a+\omega') + y \wp(a+\omega') + z\right]$$
$$\times \left[x \wp'(a+\omega+\omega') + y \wp(a+\omega+\omega') + z\right] \equiv \wp'(2a)\,\Phi(x,y,z) + \wp(2a)\,\Psi(x,y,z) + \Omega(x,y,z)$$

where Φ, Ω, Ψ are independent of a and of the fourth degree in x, y, z. Hence we may write

$$A\, x_1 x_2 \omega_1 \omega_2 \equiv \wp'(2a)\,\Phi + \wp(2a)\,\Psi + \Omega,$$

$$B\, y_1 y_2 \omega_1 \omega_2 \equiv \wp'(2b)\,\Phi + \wp(2b)\,\Psi + \Omega,$$

$$C\, z_1 z_2 \omega_1 \omega_2 \equiv \wp'(2c)\,\Phi + \wp(2c)\,\Psi + \Omega.$$

where A, B, C are suitable constants.

[illegible] the equation of [illegible] quartic) through the sixteen points of the configuration

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1. Our method is different from that of Humbert

may be written in the form

$$AU\, x_0x_2u_1u_2 + BV\, y_1y_2v_1v_2 + CW\, z_1z_2w_1w_2 = 0 .$$

quartic curve is

$$\left[U\beta'(2a) + V\beta'(21) + W\beta'(2c)\right]\Xi + \left[U\beta(2a) + V\beta(2b) + W\beta(2c)\right]\Upsilon + \left[U+V+W\right]\Omega = 0 .$$

This is identical with a given quartic of type

if

$$U\,\beta'(2a) + V\,\beta'(2b) + W\,\beta'(2c) = f ,$$

$$U\,\beta(2a) + V\,\beta(2b) + W\,\beta(2c) = g ,$$

$$U + V + W = h .$$

These equations can be solved for U, V, W only if

$$fx_0 + gy_0 + hz_0 = 0 ,$$

and this equation implies that the point T lies on a certain line L.

If T lies on this line the quantities U, V, W can be determined so that a quartic curve through the sixteen points of the Hesse configuration associated with T coincides with a given quartic curve of type (1^4), hence we

As a point T moves along a straight line the sixteen points of the Hessian configuration associated with T and Γ_3 describe a desmic quartic curve.

This desmic quartic thus appears as a mixed concomitant of Γ_3 and Dr. Morley has known of its existence as such for some time. He denotes it by the symbol C_{514}.

I have calculated the expressions for Φ, Ψ and Ω in the present case and find them to be

$$\Phi(x,y,z) = 2x\,(x^3 - g_2 y^2 z + g_3 y^3).$$

$$\Psi(x,y,z) = y(x^3 - g_2 y^2 z + g_3 y^3) - x^2\left(g_2^2 y^2 - 36\, g_3 yz + 12\, g_2 z^2\right).$$

$$\Omega(x,y,z) = \tfrac{1}{4} z\,(x^3 - g_2 y^2 z + g_3 y^3) + \tfrac{1}{2} y^2\left(g_2^2 y^2 - 36\, g_3 yz + 12\, g_2 z^2\right)$$

$$- \tfrac{1}{2} x^2 \cdot \tfrac{1}{2} g_2 g_3 y^2 - 4 g_2^2 yz + 12\, g_3 z^2 + \tfrac{1}{2} g_2^2 x^2 y$$

$$- x^4\left(g_2^3 + 24\, g_3^2\right).$$

To deduce the properties of the bitangents from the theorem which has just been proved we take the point T at one of the points of intersection of the line L with Γ_3. Let M be the tangent at T and N the other tangent from T to Γ_3. From a known property of the cubic it follows that the tangents at the remaining (other than the point of contact) [illegible]

N meets Γ_3 are the line M and the three diagonals of the quadrilateral formed by the tangents at the points where M meets Γ_3 again. The last four lines must be counted twice as belonging to the set of 6 lines and we see that the three diagonals of the quadrilateral are bitangents whose points of contact are at the corners of the quadrilateral. Since the line L meets Γ_3 in six points it follows that there are six such bisets of bitangents associated with the present method of generating the desmic [illegible]

01 02 03 , 41 42 43 , 61 62 63 ,

71 72 73 , 51 52 53 , 23 31 [illegible] .

It appears from the equation given in § 3 that two bisets of bitangents form two perspective triangles i.e. the points of intersection of corresponding pairs (01, 71), (02, 72), (03, 73) lie on a line. Now by a theorem due to Cayley[1] the points of intersection of the six pairs of bitangents [illegible]

1 Cayley's Journal Bd 43 see also H F Baker Proc London Math Soc (2) 1910

[illegible] a system of [illegible] curves [illegible] a curve [illegible] a point [illegible] curves [illegible] a point [illegible] curve [illegible] curve [illegible] a curve [illegible] curve [illegible] curve [illegible]

$$\Delta = \begin{vmatrix} & & \\ & & \\ & & \end{vmatrix}$$

$a_1' =$	$\alpha_{01}\ \beta_{01}\ \gamma_{01}$	$Fx - Gy - Hz = 0$
$b_1' =$	$\alpha_{01}\ \beta_{02}\ \gamma_{02}$	$(2gh - af)x + by + cl z = 0$
$b_2' =$	$\alpha_{02}\ \beta_{01}\ \gamma_{02}$	$af x + 2hf - by + chz = 0$
$f_1' =$	$\alpha_{01}\ \beta_{01}\ \gamma_{01}$	$af x + by + (2fg - ch)z = 0$
		$(2af - gh)x + hfy + fgz = 0$

(III)

of $a_0 a_1$; the point of intersection of $b_0 a_1$ the same as the point of intersection of $a_2' a_1'$ and so on. The complete

$$
\begin{array}{lll}
a_0 a_3 = a_0' a_3' & a_1 a_2 = b_0' c_3' & a_2 a_1 = c_0' b_3' \\
a_1 a_2 = a_0' a_2' & a_2 a_3 = b_0' c_2' & b_3 a_0 = c_0' c_2' \\
b_0 b_3 = a_0' a_1' & b_1 b_2 = b_0' b_1' & b_1 a_0 = c_0' c_1' \\
b_0 b_1 = a_0' a_3' & b_2 b_3 = b_0' b_2' & b_2 a_3 = c_0' c_3' \\
c_0 c_3 = a_0' b_3' & c_1 c_2 = b_0' b_3' & c_3 a_1 = c_0' c_2' \\
c_0 c_1 = a_0' a_2' & c_2 c_3 = b_0' b_0' & c_2 a_2 = c_0' a_0'
\end{array}
\qquad \text{(IV)}
$$

Putting $p_0 = a_0 = a_0' = b_0' = c_0'$ we have the quadratic relations

$$
\begin{array}{ll}
a_0 a_3 = a_1 a_2 & a_3 a_0 = a_2 a_1 \\
b_0 b_3 = b_1 b_2 & b_3 b_0 = b_2 b_1 \\
c_0 c_3 = c_1 c_2 & c_3 c_0 = c_2 c_1
\end{array}
$$

$$
\begin{array}{llllll}
a_0 = a_0' b_0' c_0' & a_1 = a_0' b_0' c_3' & a_2 = a_0' b_3' c_0' & a_3 = a_0' b_3' c_3' & b_0 = b_0' c_0' a_0' & b_1 = b_0' c_0' a_3' \\
b_2 = b_0' c_3' a_0' & b_3 = b_0' c_3' a_3' & c_0 = c_0' a_0' b_0' & c_1 = c_0' a_0' b_3' & c_2 = c_0' a_3' b_0' & c_3 = c_0' a_3' b_3'
\end{array}
$$

so that the node is a cusp. The equation of Γ and

$$\lambda(x-z)(y-z)^2 + \mu(x-z)^2(y-z) + \nu(x-z)^3 = 0,$$

This equation always represents a trinodal quartic with three bitangents, for if we write $X = y-z$, $Y = x-z$, $Z = x-y$ the equation becomes

$$\lambda X^2 Y^2 - Z^2) + \mu Y^2(Z^2 - X^2) = 0.$$

consequently we have the result that ∞^1 pentagons can be completely inscribed to a lemniscate. This result may also be deduced by considering the plane through three nodes.

The reciprocal configuration of the first and second was first derived by Hesse by drawing tangents to a cubic curve from the points in which it is met by a line L. If H is the Hessian of C, the curves H, C, L is a quartic curve which passes through the nine points of its configuration. This curve is the

$$\nu(\lambda-1)\,y^2z - \mu(\lambda-1)\,yz^2 + \lambda(\mu-1)\,x^2z - \nu\,\mu-1)\,xz^2 + \mu(\nu-1)\,x^2y - \lambda(\nu-1)$$

$$+ 3(\mu-\nu)\,x^2yz + 3(\nu\ \lambda)\,y^2xz + 3(\lambda\ \mu)\,z^2xy$$

Acc. di Napoli, Rend. (3) 8 (1902). Memorie di geometria, pp. 245, 246, 247.

Ann. di Matematica (3), 7 (1902), p. 279. [illegible]

It would be observed that we quoted a point which we
accepted in two quadrics and that may be quoted amongst
the different twelve but we in favor of ours do not
belong to a Hesse configuration ; for a set of 24 points
fulfilling the conditions generally depends on the arbitrary
constants whereas Hesse's configuration involves only 11.
It seems that there are at least two different types of
configurations possessing the property, for whereas in
Hesse's configuration the 9 pairs of lines containing
two quartets of points do not generally belong to a
net of conics (linear system of three terms) there is
another type of configuration differing in the
following manner in which the
Weber's pentad condition is satisfied the
configuration is marked by a quartic curve which passes
through one point of each

§ 12. A quartic curve with ∞^1 inscribed configurations of 24 points.

Going back to the working scheme of §8 we now consider the converse of Darboux's theorem that if the sides of a triangle Δ intersect the sides of a second triangle Δ' in 9 points lying on a given cubic curve H and both triangles circumscribe a conic then the six vertices of Δ and Δ' intersect on H.

The pairs of triangles Δ, Δ' inscribed in H so that the sides of Δ pass through 6 given points on H and the sides of Δ' pass through 3 other points on H are known to possess ∞^1 inscribed configurations of 24 points, viz the poles of the sides of two triangles of types Δ, Δ', and these 24 points can be arranged into two sets of three quartets possessing the properties indicated by table IV, viz any two quartets belonging to different sets lie on two straight lines.

There are 9 such pairs of lines associated with each set of 24 points. These lines all touch a curve of the third class Γ_3, the Cayleyan of the cubic, the points of intersection of the nine pairs lie on the Hessian H.

If we take two triangles whose sides intersect on H but

do not touch a conic, the pairs of them under will regard it. A cubic will possess all the properties enumerated in table IV but the 24 points will not lie on a quadric curve - the general case.

If any of the sides of the triangle Δ' lies on the conic then two of its sides coincide and the conic [illegible] at points on a line.

We will suppose that the equation of the conic is the same as in § 2. The line $x_1 = 0$ meets the Hessian in the three points and the equations of the tangents at these points are respectively

$$a_5 x_1 + a_6 x_4 = 0, \quad a_2 x_2 + a_4 x_4 = 0, \quad a_3 x_3 + a_5 x_4 = 0.$$

Now these tangents meet the Hessian again at points on the conic and so meet the sides of the triangle Δ.

The three lines whose equations are given above will form the line $x_4 = 0$ will touch the conic C_2. Six of the 8 [illegible]

are the coordinates of any point P on the quartic, the polar of P with regard to the conic, viz.

$$(ay^2 x) = 0 ,$$

must also touch C_2. Making use of the relation (5) to get rid of x_1 we find that the six lines touch a conic if the determinant

$$0 = \begin{vmatrix} a_2^2 & \cdots & \cdots & \cdots \\ a_2^2 & 0 & 0 & 0 \\ 0 & a_3^2 & 0 & a_3 a_4 \\ \cdots & \cdots & \cdots & \cdots \\ (a_2 - a_1)^2 & (a_3 - a_1)^2 & (a_2 - a_1)(a_3 - a_1) & \cdots \end{vmatrix}$$

$$a_1^2 a_4 (a_2 - a_3) \; a_2 a_3 a_4 + a_1 a_2 a_3 - a_2 a_3 a_4 - a_1 a_2 a_3 + a_1^2 a_2^2 a_3 (a_2 - a_1)(a_2 a_3 a_4 + a_1 a_2 a_3 - a_1 a_3 a_4 - a_1 a_2 a_4)$$

$$+ a_3^2 a_4 (a_3 - a_1)(a_2 a_3 a_4 + a_1 a_2 a_3 - a_2 a_3 a_4 - a_1 a_2 a_4) + a_1^2 a_2 a_3 (a_2 - a_1)(a_3 - a_1) a_4$$

$$- a_1 a_2 a_3 - a_1 a_2 a_4 - a_1 a_3 a_4 + a_1^2 a_4) + a_1 a_2 a_3 (a_2 - a_1)(a_3 - a_1) a_4 + a_2 a_3 a_4 + a_1 a_3 a_4 + a_1 a_2 a_4 - a_1^2 a_4$$

$$+ a_1 a_2 a_3 (a_2 - a_1)(a_3 - a_1)(a_2 a_3 a_4 + a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 - a_1^2 a_4)$$

$$= a_4 \{ a_1^2 [a_1^2 \cdots + a_2 a_3 (a_2 a_3 a_4 + a_1 a_2 a_3 \cdots + a_1^2 a_2^2 (\cdots$$

$$(a_2 a_3 a_4 + a_1 a_2 a_3 \cdots + a_1^2 (a_2 \cdots$$

§ 13. Diagram showing the 28 bitangents of a quartic curve and their symbols in Hesse's notation.

[illegible] curve can be regarded as the envelope of the polar conics of points on a conic Γ with regard to a cubic C_3. Now when C_3 breaks up into the pair of lines the quartic degenerates into a conic and 8 points on the conic, viz. the poles of the two lines. The [illegible] 28 bitangents of the quartic now consist of the [illegible] and the 12 lines joining pairs of the 8 points. Each of the [illegible] is represented by the symbols 01, 45 etc. [illegible] becomes the line joining the points 4, 5 when the quartic curve degenerates into the repeated conic. The numbers which are thus attached to the bitangents can be seen to be the same as those given by one of the Hessian scheme. For instance, if 0123, 4567 are the four poles of the two lines, the pairs $\begin{matrix}12 & 20 & 01 & 56 & 64 & 47\\ 03 & 13 & 23 & 47 & 57 & 67\end{matrix}$ correspond to pairs of bitangents forming a Steiner complex, as may be

seen from the method of generation of the quartic as an envelope of conics. But there are also 16 equations for 12 tangents of a Cayleyan surface in Hesse's notation. We should also notice that in the degenerate case the two tangents of the fixed pair of lines are conics in accordance with [illegible].

If we consider a hyperbolic conic which is proportional to degenerate into two straight lines, we can obtain [illegible] have just before it breaks up into a reducible conic and eight points. These do not include all the different possible types of quartic curves which can be seen by comparing the present enumeration with that of Zeuthen[1].

1. Math. Ann. Bd 7, (1874). See also Klein, Math. Ann. Bd 12. The different types of singular quartics are discussed by Miss Gentry, Dissertation, Univ. [illegible] No. 7. Bulletin, Worcester Polytechnic Institute, [illegible]

The different cases to be considered are as follows:

(1) Eight real points on the conic. The quartic curve has, before it degenerates, 28 real bitangents with real points of contact. The figure illustrates Cayley's remark that a bitangent whose [illegible]

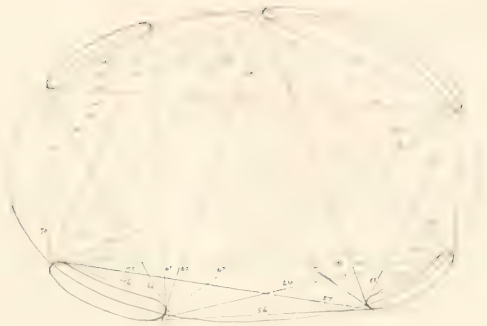
[illegible]

(2) [illegible] conic. The corresponding quartic has 16 real bitangents 15 of which have real points of contact.

(3) Four real points on the conic and two pairs of conjugate imaginary points on the conic. The quartic has 8 real bitangents 6 of which have real points of contact.

(4) Two real points on the conic and three pairs of conjugate imaginary points on the conic. The quartic has 4 real bitangents one of which has real points of contact.

(5) No real points on the conic. Four pairs of conjugate imaginary points. The quartic has 4 real bitangents each of which has imaginary points of contact.



§ 14. A property of a cubic surface with [illegible]

[illegible] angular velocity round the points A, B in [illegible]. When any line has rotated through an angle ω, all [illegible]

The transformation which arises in this way is a [illegible] with A and B as base points. In other [illegible]

[illegible] of the transformed curve, the directions of the latter being obtained by rotating those of the former "through ω" in space.

In particular, a conic through AB is transformed into a conic through AB and a circle through AB into itself. Now consider a conic through AB which touches two circles through AB. If we apply the transformation with an arbitrary value of ω we obtain a similar conic through AB touching the given circles. From this we see that a similar conic through AB

which touch the given circles by the two circles are of the same size then are called rectangular tritangents through AB having their centres [illegible]. The centres can be constructed as follows.

[illegible] the length of the tangent to the circle [illegible] Draw the locus of points [illegible] the circle [illegible] Draw the circle [illegible] Then [illegible] the centres of the required circles [illegible]

[illegible]

Now consider a cubic surface with one of its points A, B, C, D in space. A tangent plane meets the surface in the line B_1C_1 and a conic Γ. If we project on a plane L from a point V on the surface, the [illegible] from V goes to the conic [illegible]

automorphic for collineation groups G_{360} and G_{20160}.

In the present section we shall obtain a quintic variety which is in a certain sense an extension of Klein's quartic. By a natural generalisation of the sort [illegible] we are led to consider a set of 11 points A_ν [illegible]

$$\infty = \epsilon^{0}, \quad y = \epsilon^{3\nu}, \quad z = \epsilon^{9\nu}, \quad v = \epsilon^{-\nu}, \quad t = \epsilon^{-4\nu}; \qquad \epsilon^{11} = 1.$$

1 Loc. cit.

The general equation of a quadric surface through the above points is

$$Q = \lambda(x^2 - zt) + \eta(xy - zs) + \zeta(xz - yt) + \alpha(yz - xs) + \beta(y^2 - zx) + \gamma(t^2 - sx) = 0$$

$$\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial Q}{\partial z} = \frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial s} = 0$$

$$\begin{vmatrix} 2\lambda & \eta & -\zeta & \alpha & -\beta \\ -s & 2\gamma & 0 & -\alpha & 0 \\ 0 & -t & 2\lambda & 0 & -\beta \\ -\gamma & 0 & -\alpha & 2s & 0 \\ 0 & 0 & 0 & -\beta & 0 \end{vmatrix}$$

$$-2\left[\lambda^2 t^2 + \gamma^2 \beta^2 + \lambda^2 \gamma^2 + s^2 \beta^2 + t^2 \alpha^2\right] = 0$$

This variety $J = 0$ has a double point at each of the

eleven points. The equation of the tangent cone at the double point $(1,1,1,1)$ is

$$x^2+y^2+z^2+w^2+\dots+xy+xz+xw+yz+yw+zw=0$$

This cone passes through the other six points. Similarly for the tangent cones at the other six double points. Since

[illegible]

[illegible]

from the equations

$$\frac{\partial F}{\partial x}=0,\quad \frac{\partial F}{\partial y}=0,\quad \frac{\partial F}{\partial z}=0,\quad \frac{\partial F}{\partial w}=0$$

[illegible]

$$\begin{vmatrix} & & & \\ & & & \\ & & & \\ & & & \end{vmatrix}$$

Biography

S Harry Bateman was born at Manchester England on Mar 1
1882, son of Samuel Bateman and his wife Marianne
Elizabeth. I was educated at Manchester Grammar School
and proceeded from there to Trinity College Cambridge where
where I took the degrees of B.A and M.A.

After [illegible] I was [illegible] at Owens College
Manchester University 1907 10. From 1910 to
was Lecturer in Mathematics at Bryn Mawr [illegible]
and since September 1912 I have held a